Journal of Engineering Physics and Thermophysics, Vol. 83, No. 3, 2010

INFLUENCE OF THE GROUND CAPILLARITY AND OF EVAPORATION FROM THE FREE GROUNDWATER SURFACE ON FILTRATION FROM CANALS

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UDC 532.546

Mathematical models of certain groundwater flows from a canal in the presence of free-surface evaporation are considered on the basis of the two-dimensional theory of steady-state filtration. Mixed boundary-value problems of analytical-function theory are formulated and solved with the Polubarinova-Kochina method for studying such flows. Algorithms of calculation of the dimensions of the saturation zone in situations where, in filtration of water from a canal, the capillarity of its ground or the level of water in it are allowed for are based on these models. Using the obtained exact analytical dependences and numerical calculations, we made a detailed analysis of the structure and characteristic features of the modeled processes and of the influence of all physical characteristics of the models on the dimensions of the saturation zone.

Keywords: *filtration, groundwater, ground capillarity, evaporation, Polubarinova-Kochina method, Fuks linear differential equations, complex flow velocity, conformal mappings.*

Introduction. The influence of the capillary rise of a liquid on filtration from a canal (hydraulic structure) was first investigated in [1–5]. Subsequently it became clear that one could consider, on the basis of these works, only such problems in which the free surface mates directly with impermeable boundaries of the flow region. Further substantial results were associated with [6, 7] where the necessity of introducing zones where capillary water exits the earth's surface was proved, portions of the exit with evaporation and infiltration and capillary-current portions were separated, and a new analytical function for solution of the problems was proposed. This made it possible to considerably extend the class of problems of practical importance which were associated with the operation of canals, drainage headers, etc. Thus a theory of filtration in capillary grounds for any forms of external boundaries of the flow region was properly created in [6, 7]. The influence of the ground capillary in motion of water from a furrow-type irrigation canal was studied from the results obtained in [6], and a generation of this problem to the case of allowance for free-surface evaporation was given in [8, 9]. A few analytical solutions of similar problems on water filtration in a soil layer which is at a finite or infinite depth to the underlying confined horizon and from a system of canals and irrigation cannals of the furrow type were obtained in [10–15].

In all the previous investigations, consideration was given to filtration regions bounded by a water-permeable boundary on the underside; furthermore, the water level in the (irrigation) canal was assumed to be infinitesimal, so that in fact the influence of the depth of water on the character of its flow in the indicated regions was not analyzed.

In the present work, a study is made of two cases of water filtration from a canal in a ground layer of bounded thickness which is underlain by a horizontal water-confining layer in the presence of evaporation from the free groundwater surface. First the problem of groundwater flow from a canal with a small water depth in the presence of the ground capillarity is considered, and then the motion of a liquid from a rectangular canal in which there is water but no capillary rise is studied. Compared to the problems in which no consideration is given to the ground capillarity and the presence of water in the canal, in our case filtration schemes are significantly complicated by the appearance of an additional boundary portion: the streamline and the equipotential line corresponding to the capillary spread and level of water. This gives rise to an additional angular point increasing the total number of unknown parameters of conformal mapping, which occur in the process of solution of the corresponding boundary-value problem

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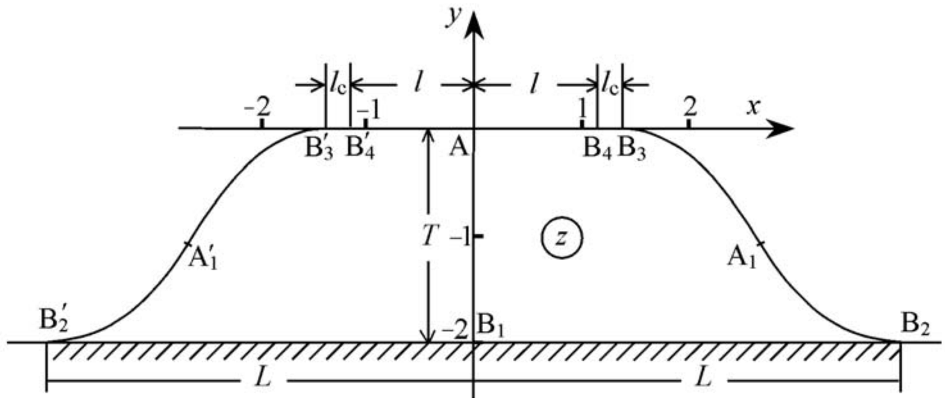

Fig. 1. Scheme of liquid flow from the canal with a small water depth for $h_c = 0.4$, $\varepsilon = 0.4$, $l = 1.2$, and $T = 2.0$.

with an unknown free boundary. To solve it we use the Polubarinova-Kochina method, which is based on the analytical theory of Fuks linear differential equations [16–19]. With the obtained exact analytical dependences and numerical calculations, the characteristic features of the modeled processes are studied with allowance for the ground capillarity, evaporation from the free groundwater surface, and the water level in the canal and the combined influence of these factors on the dimensions of the saturation zone is evaluated. The computational algorithm constructed in this manner makes it possible to judge the shape and dimensions of the zone of spread of the liquid from the canal as functions of its width and the water level in it, the ground capillarity, the intensity of free-surface evaporation of water, and the thickness of the ground stratum in each particular case. Also, the results of calculations for both schemes of motion are compared.

Filtration from the Canal with a Small Water Depth in the Presence of the Ground Capillarity. Formulation of the Problem. Figure 1 gives the scheme of plane free steady-state groundwater flow from the canal $B_4'B_4$ of width $2l$ with a small water depth to a ground layer of thickness T underlain by a horizontal water-confining layer. The flow-rate factor compensating for water filtration from the canal is a uniform free-surface evaporation of intensity ε ($0 < \varepsilon < 1$) referred to the filtration coefficient $k = \text{const}$.

We introduce a complex potential of motion $\omega = \varphi + i\psi$ and a complex coordinate $z = x + iy$ which are referred to as kT and T respectively. By virtue of the symmetry of the region of motion $B_2'A_1'B_3'B_4'AB_4B_3A_1B_2B_1$, we restrict our consideration to its right-hand half: the flow region $AB_1B_2A_1B_3B_4$.

We take $\varphi = 0$ along the canal's bottom AB_4 and $\psi = Q$ along the symmetry line AB_1. We set $\psi = Q$ on the portion B_3B_4. Thus, according to [6, 7, 17, 18], the portion B_3B_4 is considered to be an impermeable line of exit of the capillary water from the earth's surface. For the coordinate system selected in Fig. 1 and when the plane of comparison of the potentials is made coincident with the plane $y = 0$ at the boundary of the region of motion we have the following boundary conditions:

$$AB_4: \ y = 0, \ \ \varphi = 0, \ \ AB_1: \ x = 0, \ \ \psi = 0;$$

$$B_1B_2: \ y = -T, \ \ \psi = 0, \ \ B_3B_4: \ y = 0, \ \ \psi = 0;$$

$$B_2A_1B_3: \ \varphi = -y + h_c, \ \ \psi = Q - \varepsilon(x - l - l_c). \tag{1}$$

Setting $x = L$ for the portion $B_2A_1B_3$ in the second condition of (1), we obtain

$$Q = \varepsilon(L - l - l_c). \tag{2}$$

The last relation expresses the equality of the rate of flow of water from the canal to the value of its free-surface evaporation under steady-state-filtration conditions.

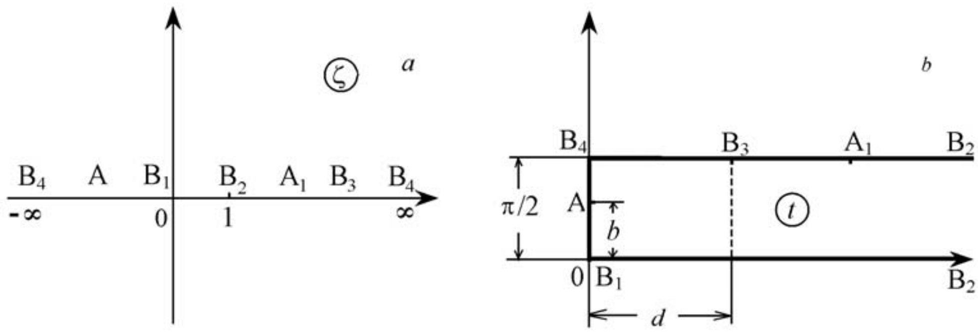

Fig. 2. Regions of the auxiliary parametric variables ζ (a) and t (b).

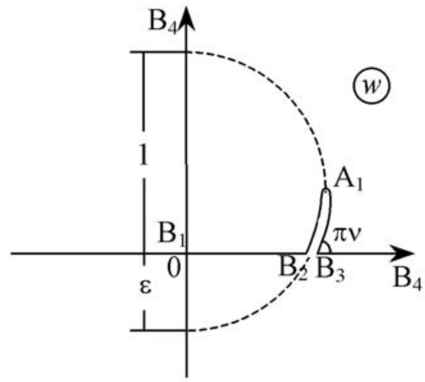

Fig. 3. Region of the complex velocity w.

We will assume that the motion of groundwater obeys Darcy's law and occurs in a homogeneous isotropic ground that is considered to be incompressible, just as the liquid filtered through it. The stratum thickness T, the canal width l, the free-surface evaporation ε, and the static height of capillary spread of water h_c are assumed to be prescribed. It is desired to determine the dimensions l_c and L of the saturation zone.

Construction of the Solution. To solve boundary-value problem (1) we use the Polubarinova-Kochina method, which is based on the analytical theory of Fuks linear differential equations. We introduce an auxiliary variable ζ and a function $z(\zeta)$ conformally mapping the upper half-plane of ζ onto the region z (the correspondence of points is indicated in Fig. 2a) as well as the derivatives

$$F = \frac{d\omega}{d\zeta}, \quad Z = \frac{dz}{d\zeta}. \tag{3}$$

For further construction, it is convenient to introduce, instead of the variable ζ, a new parametric variable t (the range of variation in t is presented in Fig. 2b), which is related to ζ by

$$t = \operatorname{artanh}\sqrt{\zeta}\,. \tag{4}$$

Let us turn to the region of the complex velocity w:

$$w = \frac{d\omega}{dz} = \frac{F}{Z}, \tag{5}$$

which correspond to boundary conditions (1) and is shown in Fig. 3. The function performing conformal mapping of the half-band of the t plane onto the region of w has been studied in [20, 21] and has the following form:

$$w\,(t) = \sqrt{\varepsilon}\,\frac{(C + \tanh t)^{\nu} \exp\,(1-\nu)\,t - (C - \tanh t)^{\nu} \exp\,(\nu - 1)\,t}{(C + \tanh t)^{\nu} \exp\,(1-\nu)\,t + (C - \tanh t)^{\nu} \exp\,(\nu - 1)\,t}, \tag{6}$$

where $C = D/\sqrt{D^2 - 1}$, $D = \cosh d$; the abscissa d is the unknown affix of point B_3 in the t plane (Fig. 2b).

Using the procedure of determination of the indices of the functions F and Z near singular points [17, 19] and taking account of expression (6), we find

$$\frac{d\omega}{dt} = \sqrt{\varepsilon}\, M \frac{(C \cosh t + \sinh t)^{\nu} \exp(1 - \nu)\, t - (C \cosh t - \sinh t)^{\nu} \exp(\nu - 1)\, t}{\Delta(t)},$$

$$\frac{dz}{dt} = M \frac{(C \cosh t + \sinh t)^{\nu} \exp(1 - \nu)\, t + (C \cosh t - \sinh t)^{\nu} \exp(\nu - 1)\, t}{\Delta(t)},$$

$$\Delta(t) = \sqrt{(\sinh^2 t + B^2)(\sinh^2 t + D^2)^{\nu}}, \tag{7}$$

where $M > 0$ and $B = \sin b$; the ordinate b is also the unknown affix of point A in the t plate (Fig. 2b). It can be checked that the functions (7) satisfy conditions (1) formulated in terms of the functions $d\omega/dt$ and dz/dt and thereby are the parametric solution of the initial boundary-value problem.

Writing representations (7) for different portions of the boundary of the t region followed by integration over the entire contour of the auxiliary region (Fig. 2b) leads to a closing of the region of motion z and thereby serves as a computation control.

This yields the expressions for the basic geometric and filtration characteristics

$$\sqrt{\varepsilon}\, M \int_0^d \Phi_{B_3B_4} dt = h_c, \quad M \int_b^{0.5\pi} X_{AB_4} dt = l, \quad M \sin\frac{\pi\nu}{2} \int_d^{\infty} Y_{B_2A_1B_3} dt = T, \tag{8}$$

which make it possible to determine the unknown parameters of conformal mapping B and D and the modeling constant M. Once the unknown constants have been found, we successively calculate the sought dimensions of the saturation zone

$$l_c = M \int_0^d X_{B_3B_4} dt, \quad L = M \int_0^{\infty} X_{B_1B_2} dt, \tag{9}$$

and then determine the flow rate Q from formula (2).

Computations can be controlled by other expressions for the flow rate Q and the thickness T:

$$Q = -\sqrt{\varepsilon}\, M \int_b^{0.5\pi} \psi_{AB_4} dt = \sqrt{\varepsilon}\, M \sin\frac{\pi\nu}{2} \int_d^{\infty} Y_{B_2A_1B_3} dt, \quad T = M \int_0^b Y_{AB_1} dt. \tag{10}$$

In formulas (8)–(10), the integrands are the expressions of the right-hand sides of (7) on the corresponding portions of the contour of the t plane.

In the limiting case where $h_c = 0$, i.e., the ground capillary is absent, in the plane of the complex velocity w, the vertex of the cut A_1 reaches the ordinate axis and the circular pentagon degenerates into a circular triangle: in the flow plane z, the inflection point A_1 merging with point B_3 reaches the abscissa axis. In this case it follows from the first formula of (8) that $l_c = 0$ for $d = 0$, $D = 1$, and $C = \infty$ and the results of [22] are obtained.

Discussion of Numerical Results. Figure 1 shows the pattern of flow from the canal with a small water depth, which is calculated for $h_c = 0.4$, $\varepsilon = 0.4$, $l = 1.2$, and $T = 2.0$. The results of calculations of the influence of the governing physical parameters h_c, ε, l, and T on the dimensions of the saturation zone L and l_c and the flow rate Q are given in Tables 1 and 2.

In each block of the tables, one of the indicated parameters is varied six times in the permissible range, whereas the remaining parameters are fixed: $h_c = 0.4$, $\varepsilon = 0.4$, $l = 1.2$, and $T = 2.0$. Figure 4a and b plots the capillary spread of water l_c (curves 1) and the width of the water spread over the water-confining layer L (curves 2) versus the parameters h_c and ε.

TABLE 1. Results of Calculations of L, l_c, and Q with Variation of h_c and ε

$h_c \cdot 10^2$	L	l_c	Q	$\varepsilon \cdot 10^2$	L	l_c	Q
10	3.461	0.0425	0.888	10	7.752	0.4998	0.605
25	3.769	0.1207	0.979	25	5.008	0.2742	0.884
40	4.031	0.2163	1.046	40	4.031	0.2163	1.046
55	4.271	0.3291	1.097	55	3.495	0.1888	1.158
60	4.3487	0.3700	1.1113	60	3.364	0.1825	1.1889

TABLE 2. Results of Calculations of L, l_c, and Q with Variation of l and T

$l \cdot 10^2$	L	l_c	Q	$T \cdot 10^2$	L	l_c	Q
30	2.326	0.1577	0.748	50	2.408	0.4245	0.313
75	3.372	0.1943	0.971	125	3.278	0.2761	0.721
120	4.031	0.2163	1.046	200	4.031	0.2163	1.046
165	4.571	0.2285	1.077	275	4.644	0.1880	1.302
180	4.7388	0.2311	1.0830	300	4.8188	0.1821	1.3747

An analysis of the data of the tables and the plots enables us to draw the following conclusions.

Increase in the canal width and in the height of vacuum due to the capillary forces in the ground and decrease in the evaporation cause the saturation zone to extend. Noteworthy is the same qualitative character of the quantities L and l_c plotted versus the parameters h_c, ε, and l. Thus, according to the data of Tables 1 and 2, the change in the height of the capillary rise of water h_c, in the evaporation ε, and in the canal width l corresponds to an increase in the spread width of 23, 130, and 104% respectively. Also, we note that the dependences of the quantity L on the characteristics h_c and l are nearly linear.

The largest influence on the width of the water spread over the water-confining layer is exerted by the free-surface evaporation: the data of Table 1 show that the value of L increases 2.3 times with decrease in the parameter ε.

Evaporation strongly influences the width of the capillary water spread, too: the same table shows that l_c grows by 174% with decrease in ε.

The right-hand block of Table 2 reflects a physically natural regularity: growth in the stratum thickness T leads to an increase in the width of the water spread over the water-confining layer and, conversely, to a decrease in the width of the capillary water spread. Thus, the width L is doubled with change in T, whereas the width l_c decreases nearly 2.3 times.

However, the most substantial influence on the value of the capillary spread l_c is exerted by the ground capillary: from the data of the left-hand block it follows that the capillary water spread increases nearly nine times with h_c.

From the two right-hand blocks of Tables 1 and 2, we note that at T = 0.5 and ε = 0.1, we obtain l_c = 0.4225 and l_c = 0.4998 respectively and consequently l_c/h_c = 1.0613 and 1.2495. Such ratios become even higher as T and ε decrease. Thus, the great importance of horizontal water suction (including that for low-capillary ground), which was first noted by N. N. Verigin [6], is confirmed.

The character of the influence of governing physical parameters on the filtration flow rate can be judged from the right-hand sections of all blocks of Tables 1 and 2: it is seen that the flow rate increases with all ε, h_c, L, and T. The most appreciable influence on the flow rate is exerted by the stratum thickness: from Table 2, it follows that the flow rate can increase 339.2%

Filtration from the Canal of a Rectangular Cross Section in the Presence of Water. Formulation of the Problem and Its Solution. Figure 5 shows the pattern of flow from a rectangular canal of width $2l$ with water level H in it ($0 \le H < T$) but in the absence of the ground capillarity (h_c = 0). Here the boundary conditions on the portions AB_1, B_1B_2, and $B_2A_1B_3$ retain their form (1), whereas the conditions on the portions AB_4 and B_3B_4 are replaced by the following ones:

$$AB_4: \; y = -H, \; \varphi = 0; \; B_3B_4: \; x = l, \; \varphi = 0. \tag{11}$$

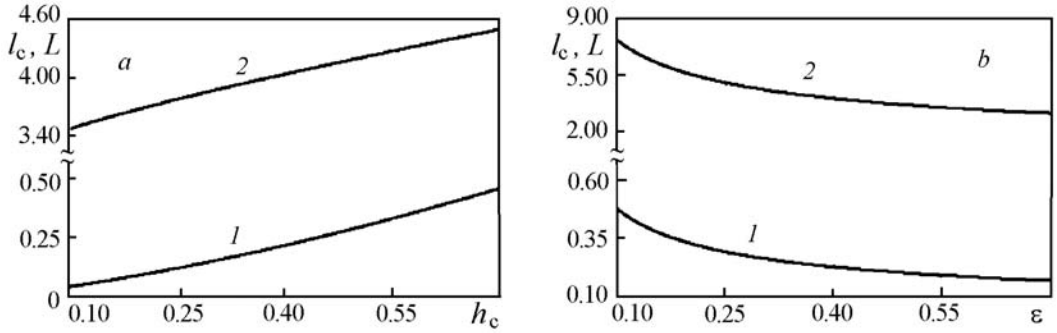

Fig. 4. Quantities l_c (1) and L (2) vs. h_c (a) for constant $\varepsilon = 0.4$, $l = 1.2$, and $T = 2.0$ and vs. ε (b) for constant $h_c = 0.4$, $l = 1.2$, and $T = 2.0$.

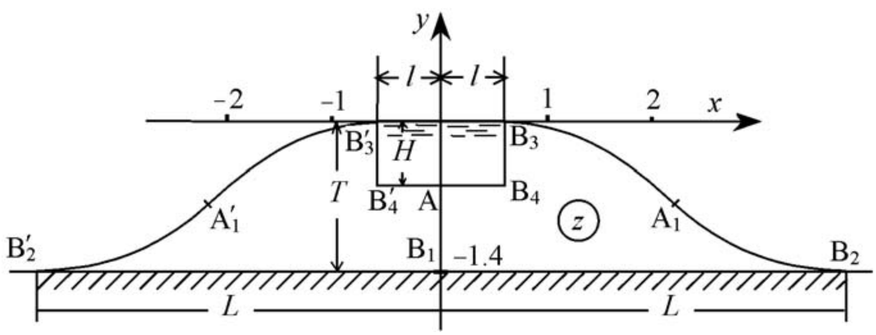

Fig. 5. Scheme of liquid flow from the canal for $H = 0.6$, $\varepsilon = 0.35$, $l = 0.6$, and $T = 1.4$.

Taking into account the coincidence of the complex-velocity region with that for the case considered earlier (Fig. 3), we obtain

$$\frac{d\omega}{dt} = \sqrt{\varepsilon}\, M \frac{(C\cosh t + \sinh t)^{\nu} \exp(1-\nu)\, t - (C\cosh t - \sinh t)^{\nu} \exp(\nu - 1)\, t}{\Delta(t)},$$

$$\frac{dz}{dt} = M \frac{(C\cosh t + \sinh t)^{\nu} \exp(1-\nu)\, t + (C\cosh t - \sinh t)^{\nu} \exp(\nu - 1)\, t}{\Delta(t)},$$

$$\Delta(t) = \sqrt{(\sinh^2 t + B^2)(\sinh^2 t + D^2)^{1+\nu}}\,. \tag{12}$$

Expressions for the model's governing parameters l, T, and L here remain as before ((8) and (9)) but the first equation of system (8) is replaced by the following equation:

$$M \int_0^d Y_{B_3B_4} dt = H\,. \tag{13}$$

Once the unknown constants of conformal mapping have been found and the water-spread width L has been determined, we calculate the flow rate from formula (2) (taking into account that $l_c = 0$). In this case computations can be controlled by the following expressions for the depth H and the flow rate Q:

$$Q = \sqrt{\varepsilon}\, M \int_d^{\infty} \Psi_{B_2A_1B_3} dt = \sqrt{\varepsilon}\, M \int_0^d (\Psi_{B_3B_4} + \Psi_{AB_4})\, dt\,, \quad H = T - M \int_0^b Y_{AB_1} dt\,. \tag{14}$$

TABLE 3. Results of Calculations of L and Q with Variation of H and ε

$H \cdot 10^2$	L	Q	$\varepsilon \cdot 10^2$	L	Q
17	2.4000	0.6299	10	4.8536	0.4254
38	2.6430	0.7250	22	3.4110	0.6184
60	2.7926	0.7674	35	2.7926	0.7674
81	2.8806	0.7982	47	2.4686	0.8782
102	2.9329	0.8165	60	2.2343	0.9806

TABLE 4. Results of Calculations of L and Q with Variation of l and T

$l \cdot 10^2$	L	Q	$T \cdot 10^3$	L	Q
17	2.2731	0.7361	617	1.6532	0.3686
38	2.5471	0.7585	1388	2.7926	0.7674
60	2.7926	0.7674	2159	3.7013	1.0854
81	3.0126	0.7709	2930	4.4022	1.3308
102	3.2269	0.7724	3702	4.9507	1.5228

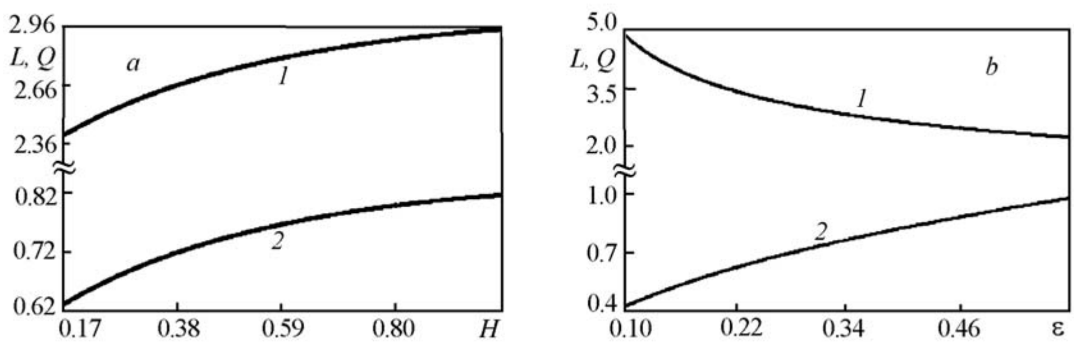

Fig. 6. Quantities L (1) and Q (2) vs. H (a) for constant $\varepsilon = 0.35$, $l = 0.6$, and $T = 1.4$ and vs. ε (b) for constant $H = 0.6$, $l = 0.6$, and $T = 1.4$.

In formulas (13)–(14), the integrands (as above) are the expressions of the right-hand sides of (12) on the corresponding portions of the contour of the t plane.

Analysis of Numerical Results and Comparison of the Results for Both Schemes. Figure 5 shows the pattern of flow from a water-filled canal, calculated for $H = 0.6$, $\varepsilon = 0.35$, $l = 0.6$, and $T = 1.4$. The results of calculations of the influence of the governing physical characteristics H, ε, l, and T on the width of groundwater spread over an impermeable bottom L and on the flow rate Q are given in Tables 3 and 4 (where the varied parameters change six times as previously). Figure 6 plots the spread width L (curves 1) and the flow rate Q (curves 2) versus H and ε.

An analysis of the data of the tables and the plots is reduced to the following.

As in the previous problem, decrease in the evaporation and increase in the stratum thickness, and in the width of the canal and the depth of water in it cause the saturation zone to extend. Evaporation exerts, as previously, a great influence on the width of filtration-water spread: the data of the right-hand block of Table 3 show that the width L increases 2.2 times with decrease in the parameter ε. However, compared to the first scheme, the water-spread width undergoes the greatest changes with variation of the stratum thickness: from the right-hand block of Table 4, it is seen that the width L grows 199.5% as the parameter T increases. The dependence of L on the canal width l turns out to be nearly linear as before, and the dependences of L and Q on the water level in the channel are qualitatively similar with equality (2) in mind.

From Table 3, it follows that variation of the parameter H leads to only slight changes in the water-spread width L and the flow rate Q (within 1.2–1.3 times), so that the influence of the water level in the canal has virtually no effect on the dimensions of the saturation zone and the filtration rate of flow from the canal.

At the same time, the character of change in the spread width L with H and ε is quite the reverse. The sections of Table 3 referring to these parameters reflect the following regularity: increase in the depth of water in the canal and decrease in the evaporation alike, contributing to the enhancement of backwater from the underlying water-impermeable bottom, cause the saturation zone to extend.

As far as the flow rate Q is concerned, the flow rate increases with all H, ε, l, and T as before; the greatest influence on Q is exerted, as previously, by the stratum thickness: from Table 4, it is seen that the change in the parameter T is accompanied by an increase of 313% in the flow rate, i.e., the situation is virtually the same as in the first scheme.

Conclusions. We have found the exact analytical solutions of the problems of water filtration from a canal to a bounded-thickness ground layer underlain by a horizontal impermeable bottom in the presence of evaporation from the free groundwater surface. The influence of such factors as the ground capillarity (for the canal with a small water depth) and the water level (for the rectangular canal) on the dimensions of the saturation zone has been studied. It has been established that decrease in the intensity of water evaporation from the free groundwater surface and increase in the thickness of the ground layer and in the width of the canal and the depth of water in it cause the saturation zone to expand. In both cases increase in all filtration characteristics is accompanied by the growth in the liquid flow rate.

The authors express their thanks to Professor S. A. Isaev for his attention during the work and for useful remarks.

NOTATION

b and d, unknown affixes (images) of points A and B_3 on the plane of the auxiliary parametric variable; B and D, unknown conformal-mapping parameters related in a corresponding manner to the affixes b and d; H, water depth in the canal; h_c, statistical height of capillary rise of groundwater; i, imaginary unit; k, filtration coefficient; L, sought width of spread of filtration water over the horizontal impermeable bottom; l, canal half-width; l_c, sought width of capillary spread of groundwater; M, scaling constant of modeling; Q, sought filtration rate of flow from the canal; T, depth of the horizontal impermeable bottom; t, auxiliary parametric variable; w, complex flow velocity; x, y, z, abscissa, ordinate, and complex coordinate respectively of a point of the flow region; ε, intensity of free-surface evaporation; ζ, upper half-plane; ν, angle (in fractions of π) at the vertex B_3 of the complex-velocity region; φ, velocity potential; ψ, stream function; ω, complex flow potential. Subscripts: c, capillary.

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